1 Basic two-equation model

As a basic RANS model, the $k-\varepsilon$ model

1.1 Conservation form of the system

The Reynoldsed Navier-Stokes system relying to the $k-\varepsilon$ model is written in a conservative form as

$$\frac{\partial Q}{\partial t}+\frac{\partial F(Q)}{\partial x}+\frac{\partial G(Q)}{\partial y}= \frac{1}{\mathrm{Re}}\left(\frac{\partial R(Q)}{\partial x}+\frac{\partial S(Q)}{\partial y}\right)+\frac{\partial R^{turb}(Q)}{\partial x}+\frac{\partial S^{turb}(Q)}{\partial y}+\Omega(Q)$$

where :

- $Q(x,y,t)=(\rho,\ \rho u,\ \rho v,\ E^{'},\ \rho k, \rho\varepsilon)^T$ are the nondimensionalised conservative variables.
- $F(Q)$ and $G(Q)$ are the convective flux functions.
- $R(Q)$, $S(Q)$ are the laminar viscous fluxes. Re is the laminar Reynolds number obtained at the nondimensionalisation.
- $R^{turb}(Q)$, $S^{turb}(Q)$ are turbulent viscous flux functions.
- $\Omega(Q)$ is the source term related to the $k-\varepsilon$ model.

Viscous turbulent stress also involves a diagonal term $2/3\rho k\mathcal{I}_d$ ($\mathcal{I}_d$ is the identity matrix) that is accounted through an adhoc variable change:

$$\begin{cases} p^{'}=p+\dfrac{2}{3}\rho k \\ E^{'}=E+\beta\rho k \quad \text{where} \quad \beta=-1+\dfrac{2}{3(\gamma-1)} \end{cases}$$

with

$$\begin{cases} p=(\gamma-1)\rho C_v T \\ E=\rho C_v T+\dfrac{1}{2}\rho(u^2+v^2)+\rho k \end{cases}$$

where p is the pressure, E— the total energy per volume unit, ρ— the density, k— the turbulent kinetic energy, C_v holds of the specific heat for constant volume, T — the temperature, γ — the specific heat ratio assumed as constant ($\gamma = 1.4$ for a perfect gas) and u, v are mean flow velocity components. The relation between $E^{'}$ and $p^{'}$ is described by

$$p^{'} = (\gamma - 1)\left(E^{'} - \frac{1}{2}\rho\left(u^2 + v^2\right)\right)$$

Then convective fluxes turn to be:

$$F(Q) = \begin{pmatrix} \rho u \\ \rho u^2 + p' \\ \rho uv \\ (E' + p')\, u \\ \rho uk \\ \rho u\varepsilon \end{pmatrix}, \quad G(Q) = \begin{pmatrix} \rho v \\ \rho uv \\ \rho v^2 + p' \\ (E' + p')\, v = \\ \rho vk \\ \rho v\varepsilon \end{pmatrix}.$$

Laminar viscous fluxes are written as

$$R(Q) = \begin{pmatrix} 0 \\ \tau_{xx} \\ \tau_{xy} \\ u\,\tau_{xx} + v\,\tau_{xy} + \dfrac{\gamma\,\mu}{\mathrm{Pr}}\dfrac{\partial e}{\partial x} + \beta\mu\,\dfrac{\partial k}{\partial x} \\ \mu\,\dfrac{\partial k}{\partial x} \\ \mu\,\dfrac{\partial \varepsilon}{\partial x} \end{pmatrix},$$

$$
S\left(Q\right)=\begin{pmatrix} 0 \\ \\ \tau_{xy} \\ \\ \tau_{yy} \\ u\,\tau_{xy}+v\,\tau_{yy}+\dfrac{\gamma\,\mu}{\mathrm{Pr}}\dfrac{\partial e}{\partial y}+\beta\mu\,\dfrac{\partial k}{\partial y} \\ \mu\,\dfrac{\partial k}{\partial y} \\ \mu\,\dfrac{\partial \varepsilon}{\partial y} \end{pmatrix}.
$$

and turbulent viscous fluxes are defined as follows

$$
R^{turb}\left(Q\right)=\begin{pmatrix} 0 \\ \\ \tau^t_{xx} \\ \\ \tau^t_{xy} \\ u\,\tau^t_{xx}+v\,\tau^t_{xy}+\dfrac{\gamma\,\mu_t}{\mathrm{Pr}_t}\dfrac{\partial e}{\partial x}+\dfrac{\mu_t}{\sigma_k}\dfrac{\partial k}{\partial x}+(1+\beta)\dfrac{\mu_t}{\sigma_k}\dfrac{\partial k}{\partial x} \\ \dfrac{\mu_t}{\sigma_k}\dfrac{\partial k}{\partial x} \\ \dfrac{\mu_t}{\sigma_\varepsilon}\dfrac{\partial \varepsilon}{\partial x} \end{pmatrix},
$$

$$
S^{turb}(Q) = \begin{pmatrix} 0 \\ \tau^t_{xy} \\ \tau^t_{yy} \\ u\,\tau^t_{xy} + v\,\tau^t_{yy} + \dfrac{\gamma\,\mu_t}{\mathrm{Pr}_t}\dfrac{\partial e}{\partial y} + \dfrac{\mu_t}{\sigma_k}\dfrac{\partial k}{\partial y} + (1+\beta)\dfrac{\mu_t}{\sigma_k}\dfrac{\partial k}{\partial y} \\ \dfrac{\mu_t}{\sigma_k}\dfrac{\partial k}{\partial y} \\ \dfrac{\mu_t}{\sigma_\varepsilon}\dfrac{\partial \varepsilon}{\partial y} \end{pmatrix},
$$

where τ_{ij} and τ^t_{ij} represents respectively the laminar and turbulent stress tensors which are given by

$$
\begin{aligned}
\tau_{ij} &= \mu\left(\frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i}\right) - \frac{2}{3}\mu\frac{\partial u_k}{\partial x_k}\delta_{ij} \quad , \quad u_1 = u \; , \; u_2 = v \\
\tau^t_{ij} &= \mu_t\left(\frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i}\right) - \frac{2}{3}\mu_t\frac{\partial u_k}{\partial x_k}\delta_{ij}
\end{aligned}
$$

Coefficient of turbulent viscosity μ_t is found from relation

$$
\mu_t = c_\mu \frac{\rho k^2}{\varepsilon}
$$

The variation of nondimensional laminar viscosity coefficient μ as a function of a dimensional temperature T is defined by the Sutherland law

$$
\begin{cases}
\mu(T) = \mu_{ref}\dfrac{T}{T_{ref}} & \text{if } T \leq 120\ K \\
\mu(T) = \mu(120)\left(\dfrac{T}{120}\right)^{1.5}\left(\dfrac{120+110}{T+110}\right) & \text{if } T \geq 120\ K
\end{cases} \tag{1}
$$

The source terms are

$$
\Omega(W) \quad = \quad \begin{pmatrix} 0 \\ 0 \\ 0 \\ \beta\omega_k \\ \omega_k \\ \omega_\varepsilon \end{pmatrix}
$$

The nondimensional parameters $\Pr = 0.725$, $\Pr_t = 0.86$ are respectively the laminar and turbulent Prandtl numbers and Reynolds number is $\text{Re} = \rho_{ref} u_{ref} L_{ref} / \mu_{ref}$. Notations ρ_{ref}, u_{ref}, L_{ref} and μ_{ref} hold respectively for a reference density, velocity, length, viscosity. Finally, it is set

$$\begin{cases} \omega_k = -\rho\,\varepsilon + \mathcal{P} \\ \omega_\varepsilon = c_{\varepsilon 1} \dfrac{\varepsilon}{k} \mathcal{P} - c_{\varepsilon 2} \dfrac{\rho\,\varepsilon^2}{k} \end{cases}$$

where $\mathcal{P}$ denotes the production term of the turbulent kinetic energy which is given

$$\mathcal{P} = -\left(\frac{2}{3}\rho k \delta_{ij} - \mu_t \left(\frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i} - \frac{2}{3}\frac{\partial u_k}{\partial x_k}\delta_{ij}\right)\right)\frac{\partial u_i}{\partial x_j}$$

Constants c_μ, c_{ε_1}, c_{ε_2} empirically defined from experiments and are equal $c_\mu = 0.09$, $\sigma_k = 1$, $c_{\varepsilon_1} = 1.44$, $c_{\varepsilon_2} = 1.92$